Coupled Dynamics in a Phillips Machine Model of the Macroeconomy

by Stefano Zambelli

1. Introduction

Phillips (1950, p. 283) stated that his analog machine was «... an attempt to develop some mechanical models which may help non-mathematicians by enabling them see the quantitative changes that occur in an inter-related system of variables following initial changes in one or more of them».

Here it is claimed that it was much more than just that. From the theoretical as well as practical point of view, simulations are important not only as *pedagogical* devices, but are also essential in order to study non trivial problems for which precise and/or analytical solutions are not known or knowable. This is almost always the case and hence the use of theoretical as well as analogue means for computing are indispensable.

Mathematics – any type one can endorse to – has the limitation that it has to be, in a way or the other, tractable (at least from a symbolic point of view). Once a researcher uses mathematics or meta-mathematics, clarity of the methods and of the aims may be obtained, but he may end up leaving out details or «modify» the problems so as to fit the tight jacket of the mathematical notation. The final result may be something far away from the problem which was originally at hand.

When Phillips constructed his machines he had in mind the possibility of linking the mathematical representation of some theoretical economic mechanisms with an analogue machine, closely reproducing the theoretical mechanisms. Some reflections on the actual functioning of the machines – with all its physical constraints – would convince most readers that his machines are not just pedagogical devices. They are also useful constructions that generate data that should have been studied by Phillips himself, for understanding

This paper has been written, thanks to the valuable suggestions and the inspirations of my teacher, colleague and friend Vela Velupillai. Just as an example on how he has been influential in shaping this article, let me just say that the title is his suggestion. I am grateful for it. I would also like to thank Selda Kao and V. Ragupathy for their indispensable help.

the tenability of the (often linear) mathematical formulations that the Phillips machine itself was meant to represent. It should have been a type of laboratory to be used to run experiments and not only as a pedagogical tool for *non-mathematicians*. In fact, the more famous Phillips curve is a result that came out of the attempts to have the MONIAC provide «answers» to the questions posed to it.

Clearly, the Phillips Machine was a remarkable project and was extremely advanced. At the time when Phillips was constructing the machine, others were building the very first electronic digital computers. It was the time when one of the first computers, the ENIAC, was operating with vacuum tubes and its memory could store only 20 numbers with 10 digits.

The idea of discussing economic problems by making an analog(y) with other systems is, of course, not new. Following the history of economic thought one can find many examples. Irving Fischer, a great *mathematical* economist, used the analogies of rational mechanics and that of hydraulic systems. In fact, he did construct an analog hydraulic machine. And also Joseph Schumpeter, one of the greatest *non-mathematical* economists of the 20^{th} century, used a large number of analogies with the aim of trying to encapsulate his ideas on business cycles. These systems went from «hydraulic pendula», «sound waves» and so on. He was demanding the «mathematically» oriented economists to develop models to tackle his «economics». The exchange he had with Frisch (1933) is a demonstration of this aim and does contain a description of his many conceptual metaphors to analog model[1].

2. The Linearity Dogma

The construction of a machine to be used for studying the economy and stabilization policy is, no doubt, a great achievement. But Phillips, with the exceptions of a few (among them Goodwin and Hicks), did fell into what Samuelson (1974, p. 10) called a dogma. Samuelson writes

> ... Thus, by 1940, Metzler and I as graduate students at Harvard *fell into the dogma* that all economic business-cycle models should have damped roots. We accepted Frisch's criticism of the Kalecki procedure of imposing constraints on his parameter-estimating equations so that roots would be neither damped nor undamped... what was so bad about the dogma? Well, it slowed down our recognition of the importance of *non-linear* auto-relaxation models of the van der Pol-Rayleigh type, with their characteristic amplitude features lacked by linear systems. And, in my case, it led to suppressing development of the Harrod-Domar exponential growth

[1] Please note that both Fisher and Schumpeter have been the founders of the Econometric Society. They were searching for ways to model their ideas with the use of mathematical tools. I think that it can be claimed that both Fisher and Schumpeter were not «modifying» their theories at all so as to adjust the available mathematical tools.

aspects that kept thrusting themselves on anyone who worked with accelerator-multiplier systems (emphasis added).

Velupillai (1992) has made a strong point, which is easily supported with a careful reading of the original article, that Frisch's (1933) «Propagation Mechanism» was, as explicitly stated by Frisch himself, a non-linear one. It is in attempting to allow for an explicit solution, that Frisch made the functional forms linear as *first approximations*. These *first approximations* were never removed and many students of the business cycles, following Frisch's approach, fell into the dogma of linearity. The mathematical business cycle models by Kalecki (1935), Samuelson (1939), Metzler (1941) are all linear and the (damped or undamped) cycles are possible, thanks to leads and lags of the variables. Kaldor (1940) did present a non-linear model of the cycle using diagrams, but did not have a mathematical formulation for it. Literary economists, like Schumpeter, had to rely on their written explanations to be able to demonstrate the non-linear elements of their theories. As Samuelson pointed out, most mathematically oriented economists, including him, fell into the dogma of linearity and his famous *multiplier-accelerator* model is an example of this.

Economics is replete with nonlinear relations. One can claim that there is no actual economic phenomenon or magnitude which is not nonlinear[2]. The very existence of physical constraints should be enough to convince any reader.

As did almost all economists of his time, Phillips also fell into the linearity dogma in several ways. First, there is no doubt, at least to this writer, that the MONIAC was and is a nonlinear machine. It is nonlinear simply because there is a maximum speed in which a flow of water can fall. This is so in general, i.e. when the water is not constrained inside a pipe, but it is even more so when constrained inside a pipe. If the water falls through a pipe from one tank to another located below, the speed at which the water would flow would depend on the pressure (quantity of water in the tanks) and would change as the water decreases from one tank and increases in the other. Moreover, the pipes do bend and each bending is a form of nonlinearity, which introduces different frictions at the bending regions. Furthermore, there are «ceilings» and «floors» that limit the operation of the system.

Second, the mathematical presentation of the model made by Phillips is

[2] One should not even feel the need to explain. But given that economic theory is filled with linear relations, we are now so accustomed to think linearly that I feel obliged to spend a few words on it. Just think in terms of physical constraint. A machine which functions by transforming factors of production into the final output will be naturally bound to a maximum speed, beyond which it will either break or produce less. The consumption capacity of individuals is limited by satiation, i.e. for each unit of time there is only so much that one individual can consume. The speed at which goods can be transferred from one place to another is limited by a maximum speed. Any aggregate economy cannot use more natural resources, land and labour than it actually has. And so on. Obviously these constraints can be formally captured only with non-linear functional forms.

constrained to linear functional forms. It is clear that he was mostly inspired by Metzler's (1941) linear leads and lags model[3].

Third, it will be claimed below that the different mathematical models presented in his Phillips (1950) article are all linear difference-differential equations. The difficulty in solving a difference-differential equations model is that, its solution needs, as «initial condition», the values associated to the interval $[(t_0 - \varepsilon), t_0]$, where ε is the length of the lag. Given the difficulties to deal with the specification of this function, which would be most likely non-linear, further simplifying assumptions are made. One of them is to consider the lag ε constant and the other is to assume that any function mapping the values from $[(t_0 - \varepsilon), t_0]$ to $[t_0, (t_0 + \varepsilon)]$ is a constant function (i.e. linear).

3. The Phillips Linear Difference-Differential Equations

Phillips (1950) article contains the description of analogue machines which are graphically described in terms of four figures. One is an actual photograph of the first machine and the other three are drawings of their functioning. As mentioned above, an inspection of these three drawings would show that the machine could not work in a linear fashion: it could do so only when it is «forced» to do so.

Phillips describes the Phillips machine for different cases:

a) *The Multiplier with Constant Rate of Interest* (Phillips, 1950, pp. 294-6);

b) *The Multiplier with Constant Quantity of Money* (Phillips, 1950, pp. 296-7);

c) *The* (Linear) *Accelerator* (Phillips, 1950, pp. 297-8).

For the above cases, the mathematical description is highly linear and it makes rather strong assumptions.

3.1. *a*) *The Multiplier with Constant Rate of Interest (Phillips, 1950, pp. 294-6) – A comment*

The model described by Phillips assumes a constant period of production P, and it does imply constant consumption. These assumptions are very strong. Phillips writes, with the meaning given in modern literature on national accounting, $Y = C + S$, $E = C + I$ and $E - Y = dM_1/dt$. Given that $I(t) \equiv S(t)$ is an accounting identity, in order for equation

[3] When Phillips refers to Hicks' work he refers to his assumption about leads and lags, and the distinction between liquid and working stocks (see Phillips, 1950, p. 289). When he refers to Goodwin's work, only to support a linear relation between income and transactions money balances (Phillips, 1950, p. 289) or in the case of the use of a *linear* accelerator (Phillips, 1950, p. 298) (this is not clear). But, he does not use Hicks-Goodwin flexible accelerator in a substantial manner in his description of the Phillips Machine.

(1) $$I - S = \frac{dM_1}{dt},$$

in Phillips (1950, p. 294) to hold, it must be that the savings value is lagged (or the other way about) with respect to investment[4].

The equation

(2) $$M_1 = PY \quad \text{or} \quad dM_1 = PdY$$

in Phillips (1950, 294), is only apparently unproblematic. The assumption of a constant period of production which underlies this equation is a very strong assumption. Without this, the correct equation for the variations would be $dM_1 = YdP + PdY$.

In order to allow consistency between (1) and (2), we must have that the change in the quantity of money, M_1, has to be a change in the quantity of monetary units with a lag L, $I(t) - S(t - L) = \frac{dM_1}{dt}$, and hence we have that $E(t) - Y(t - L) = I(t) - S(t - L) = P\frac{dY(t)}{dt} = \frac{dM_1}{dt}$.

These relations imply a difference-differential system of equations or a nonlinear system (or both). Moreover an inspection of the equations will also determine that $C(t) = C(t - L) = \overline{C}$. This requires a very special and specific dynamical path and highly specific functional forms. In fact, the description provided at the top of page 296 is equivalent to a mixed system of difference-differential equations. An inspection of the drawing of the machine, Figure 3 (Phillips, 1950, p. 290), will confirm that there is a lag between the *S* and *I*. Clearly, whether water molecules become consumption or saving would depend on the different valves and the liquid in the different tanks, but saving is «transformed» into investment (or income *Y* is transformed into demand *E*) only after it has gone through the pipes going from the junction *Y-C-S* down to the junction *C-I-E* and this takes time.

And one should not forget the «multiplier tradition» which Phillips encapsulates with his equation

(3) $$S(t) = \sigma Y(t)$$

Where, σ is the marginal propensity to save (and $1/\sigma$ becomes the multiplier). The equilibrium condition $E = Y$ does require that $Y^{Eq} = (1/\sigma)I^*$, but

[4] Clearly, under correct national accounting, it can be argued that the very existence of a positive investment $I(t)$ which is equal with $S(t)$ determines a change in the stocks of the financial magnitudes of at least some economic actors. In the functioning of the highly simplified economic system that the Phillips machines represent, the only financial asset we have is money, therefore, the following relation has to hold $I(t) \equiv S(t) \equiv \Delta M_1/\Delta t \approx dM_1/dt$ and hence $I(t) - S(t) \equiv \Delta M_1/\Delta t = 0$ is the only case possible.

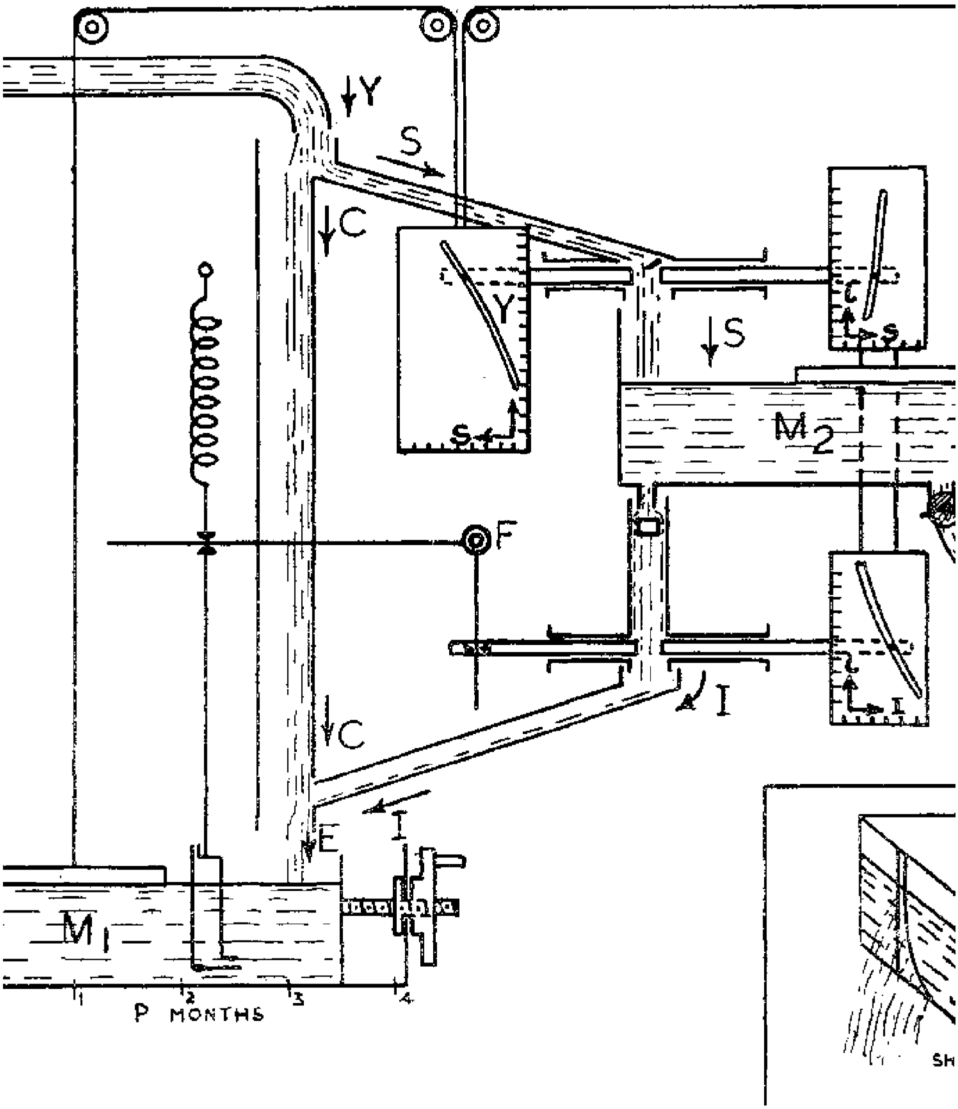


Fig. 1

the dynamics towards the equilibrium is quite another thing. It is here that Phillips, in his attempt to control too many determining factors, does expose his reasoning to the risks of confusing the causality of the determination of the levels.

3.2. *b) The Multiplier with Constant Quantity of Money (Phillips, 1950, pp. 296-7) – A comment*

The equations of this section are based on the equations of section 3.1 for the case in which the money supply M_1 (and M_2) is assumed to be constant. The well known issue of whether S determines I or I determines S is not to be taken lightly here. Equation (3) above is a definition of Savings, which can only be a fraction of the income not spent in consumption. But if it is a description of the decision of consumption, saving is residual or if it is a description of the level of saving, it is consumption which is residual. Phillips does not mean that equation (3) is residual and when considering the case with constant quantity of money, the functional form describing the saving decision is given by:

(4) $$S = \sigma Y + \xi i$$

where, ξ is the slope of the interest-savings curve. Moreover, the quantity of money held for transaction purposes is assumed to be constant here. This adds a further constraint to the model. In order to fulfil these constraints, these other variables would have to follow very specific behavioural rules. From equation (2) we have also that $\dot{M}_1 = P\dot{Y}$ and the total money supply is given by $M = M_1 + M_2$ where M_2 are *idle or surplus balances*. If M has to be kept constant and if $M_1 > 0$ and $M_2 > 0$, it is obvious that the functional forms describing the determinants of the output Y have to be nonlinear. This is in sharp contrast to what assumed by Phillips in the article.

3.3. *c) The (Linear) Accelerator (Phillips, 1950, pp. 297-8) – A comment*

The description of the accelerator in this section is highly linear. Investment is given by (Phillips, 1950, 298)

(5) $$\frac{dI}{dt} = \frac{\beta}{\gamma}\frac{dY}{dt} - \frac{I}{\gamma}$$

And the multiplier is a linear function with a delay factor: $I(t) = P\frac{dY(t)}{dt} + \sigma Y(t)$.

Differentiation of (5), combined with the multiplier, leads to the second order linear differential equation (which is equation 15 in Phillips 1950, p. 298)

(6) $$\gamma P\frac{d^2Y(t)}{dt^2} + (\gamma\sigma + P - \beta)\frac{dY(t)}{dt} + \sigma Y(t) = 0$$

4. Nonlinear Phillips Machine and the Flexible Accelerator

The mathematical descriptions of the functioning of the MONIAC made by Phillips himself are linear. But, as claimed above, the consistency with Phillips own assumptions would require acknowledging the presence of nonlinearities. Moreover, the actual Phillips machine is intrinsically nonlinear. The speed in which water flows between the different tanks is a nonlinear function of the input and it depends on the size and curvatures of the tubes and so on.

Another very important source of nonlinearity is the one that is encapsulated in the accelerator. The contributions by Hicks and Goodwin have shown the importance of this concept. According to them, eq. 5 above is actually non linear in the sense that there are boundaries (a ceiling and a floor) which limit the speed of accumulation and decumulation of capital. This implies, almost tautologically, that the model is nonlinear. If this is so for their paper and pencil «theoretical» model, it must be even more so for the Phillips machine, where the physical constraints are set by physical boundaries such as fixed maximum volume, gravitational force and so on.

The Phillips machine is, in my view, a special case of a more general accelerator relation

$$I(t) = \dot{K}(t) = F(Y(t), \dot{Y}(t)) \tag{7}$$

With the standard meaning for the symbols, $I(t)$ is aggregate investment, $K(t)$ is aggregate capital and $Y(t)$ is aggregate output.

A reasonable functional form which is consistent with Goodwin's flexible accelerator idea may be given by[5]:

$$\begin{gathered} I(t) = \dot{K}(t) = \varphi(vY(t) - K(t)) \\ \frac{\delta\varphi(vY(t) - K(t))}{\delta(vY(t) - K(t))} > 0 \quad \text{and} \quad \frac{\delta\varphi(0)}{\delta(vY(t) - K(t))} = v \\ \lim_{vY(t) - K(t) \to +\infty} \varphi(v(Y(t) - K(t)) = k^{**}; \quad \lim_{vY(t) - K(t) \to -\infty} \varphi(v(Y(t) - K(t)) = k^{*} \end{gathered} \tag{8}$$

[5] For further elaborations and explanations, see Zambelli (2011). The functional form presented in the text has an argument which is different from both Hicks (1950) and Goodwin (1951) accelerator models. The seminal works by Frisch (1933), Hicks (1950) and Goodwin (1951) contain a minor infelicity. The idea of the accelerator is the idea that investment is determined by the difference between the *desired* level of capital and the *actual* level of capital. The infelicity is given by the fact that a dynamic equation for $K(t)$ is not made explicit (although mentioned in the textual description) and is actually removed. Any attempt to reinsert will show that in all the three models while $Y(t)$ stays in between boundaries (like the constant in the computation of the primitive function of an integral) $K(t)$ will evolve without boundaries. Here, this is amended and the difference between desired capital, $vY(t)$, and actual capital, $K(t)$, are the determinants of investment decisions. This explains the formulation of the text.

The idea behind the above functional form is similar to that of the flexible accelerator of the original article by Goodwin (1951), where $vY(t)$ is the desired capital level, and $K(t)$ the existing one. Under normal conditions, i.e., near normal production capacity exploitation, the net investment adjusts fast to the production needs, so that around the equilibrium condition, i.e., $vY(t) - K(t) = 0$, a linear accelerator holds. In fact, $\delta\varphi(0)/\delta Y(t)$ is equal to the constant capital-output ratio v. When current demand is inadequate with respect to the production capacity, either because it is too high or too low, the investment levels tend to either k^{**} or k^{*}: the production or destruction of capital goods per unit time cannot go above or below these physical limits. It is the opinion of this writer that the water flowing inside the Phillips machine has to follow similar constraints.

Like in Phillips (1950), we can work by assuming two different lags: the lag between consumption decision and its actual realization, ε (Robertson lag); the lag between the moment in which decision of investment is made and its realization, $\vartheta + \varepsilon$ (Lundberg lag).

The total demand at time $t + \vartheta + \varepsilon$ is given by

$$Y((t+\vartheta)+\varepsilon) = C((t+\vartheta)+\varepsilon) + I((t+\vartheta)+\varepsilon) \tag{9}$$

Where, ε is the Robertson lag and $\vartheta + \varepsilon$ the Lundberg lag.

The lag between the moment in which income is earned and the time at which it is spent may be described as follows:

$$C((t+\vartheta)+\varepsilon) = C_0 + Y(t+\vartheta) \tag{10}$$

The fact that it takes *time to build*, and consequently, there is a lag between the moment at which an investment decision is made and time of delivery of the capital goods, may be described as follows:

$$I((t+\vartheta)+\varepsilon) = \dot{K}(t) = \varphi(vY(t) - K(t)) \tag{11}$$

Substituting equations, (10) and (11) into (9) we have the law of motion of our economy in the form of a mixed nonlinear difference-differential equation. This differential equation can be «solved» either through digital computation or through an analog machine, like the Phillips machine. It is my conjecture that the Phillips machine must have constraints represented by the equation above.

Following the method used by Goodwin (1951), in order to maintain the structure of the model as simple as possible, the mixed difference-differential equation can, with some loss in precision, be approximated by a second order differential equation, using Taylor series expansion. Doing so, following Goodwin(1951)we obtain

$$\varepsilon\vartheta\ddot{Y}(t) = -(\varepsilon + (1-c)\vartheta)\dot{Y}(t) - (1-c)Y(t) + C_0 + \varphi(vY(t) - K(t)) \tag{12}$$

The state space representation of equation (12) is given by:

(13a) $$\dot{Y}(t) = Z(t)$$

(13b) $$\dot{Z}(t) = b\ [Co - (l - c)\ Y(t) - aZ(t) + \varphi(vY(t) - K(t))]$$

(13c) $$\dot{K}(t) = \varphi(vY(t) - K(t))$$
$$b = \frac{1}{\varepsilon\theta};\quad a = \varepsilon + \vartheta(1 - c)$$

One can check that the above model can account for cyclical behavior. For a wide range of the parameter values, the dynamical system evolves towards a limit cycle. In fact the analysis of the Jacobian shows that the equilibrium point is a repeller and that the variables evolve inside a closed compact set.

5. Coupling or Coupled Phillips Machines: The Two Nation System Case

Phillips (1950, p. 305) concluded his article as follows:

> It is possible to connect together two of the models shown in Figure 4, to deal with the multiplier relationship between the incomes of two countries, or of one country and the rest of the world. To connect more than two would be difficult, since each country would have to have a propensity to import function for each other country. The easiest method of interconnection would be to assume a fixed rate of exchange, and run the imports flow of one model into the export tube of the other.

This coupling of machines is definitely worthwhile, if the machine is not a linear machine. In the case of a linear machine, the results are relatively simple and analytical solutions can be computed. This is not so in the case of a nonlinear machine. From the arguments presented earlier in the paper, it should be clear that it is my conjecture that the Phillips machine was and is a nonlinear one. The model presented in section 4 is most likely the closest one to the Phillips machine. It describes the case in which Phillips machine operates with fixed interest rates. This being the case, a digital coupling of two digital would-be Phillips machines has been made in Zambelli (2011). As described above in the Phillips' (1950) quotation, the countries are linked through trade where the exchange rate is fixed and where it is the case that «the imports flow of one model (run) into the export tube of the other».

A natural way to introduce coupling is by considering the fact that countries are linked through trade.

The descriptions of national economic accounts flows of the countries 1 and 2 are given by:

$$IM_1(t+\vartheta_1+\varepsilon_1)+Y_1(t+\vartheta_1+\varepsilon_1)\equiv C_1(t+\vartheta_1+\varepsilon_1)+ \\ +I_1(t+\vartheta_1+\varepsilon_1)+X_1(t+\vartheta_1+\varepsilon_1)$$

(14)

$$IM_2(t+\vartheta_2+\varepsilon_2)+Y_2(t+\vartheta_2+\varepsilon_2)\equiv C_2(t+\vartheta_2+\varepsilon_2)+ \\ +I_2(t+\vartheta_2+\varepsilon_2)+X_2(t+\vartheta_2+\varepsilon_2)$$

Where, $IM_{1,2}$ are imports of countries 1 and 2. Following traditional lines, one can assume that the demand for foreign goods is the demand for final goods, which, thereby, is also exposed to the same lag effects as that of consumption mentioned earlier. Therefore:

$$IM_1(t+\vartheta_1+\varepsilon_1)=m_1Y_1(t+\vartheta_1)$$

(15)

$$IM_2(t+\vartheta_2+\varepsilon_2)=m_2Y_2(t+\vartheta_2)$$

Obviously the export of one country is the import of the other, so that:

$$X_1(t+\vartheta_1+\varepsilon_1)=m_2Y_2(t+\vartheta_1+\varepsilon_1-\varepsilon_2)$$

(16)

$$X_2(t+\vartheta_2+\varepsilon_2)=m_1Y_1(t+\vartheta_2+\varepsilon_2-\varepsilon_1)$$

Maintaining the model described by system (eq. 13), enlarged by (14) (15) and (16), and proceeding through a Taylor series expansion approximation, we obtain:

(17a) $$\dot{Y}_1(t)=Z_1(t)$$

(17b) $$\dot{Z}_1(t)=\omega\left(\Omega_1(t)+m_1\frac{(\varepsilon_1-\varepsilon_2)}{\varepsilon_1}\Omega_2(t)\right)$$

(17c) $$\dot{K}_1(t)=\varphi(v_1Y_1(t)-K_1(t))$$

(17d) $$\dot{Y}_2(t)=Z_2(t)$$

(17e) $$\dot{Z}_2(t)=\omega\left(\Omega_2(t)+m_2\frac{(\varepsilon_2-\varepsilon_1)}{\varepsilon_2}\Omega_1(t)\right)$$

(17f) $$\dot{K}_2(t)=\phi(v_2Y_2(t)-K_2(t))$$

where:

$$\Omega_1(t)=\frac{C_{01}-e_1Y_1(t)-a_1Z_1(t)+m_2(Y_2(t)+f_1Z_2(t))+\varphi_1(v_1Y_1(t)-K_1(t))}{\vartheta_1\varepsilon_1}$$

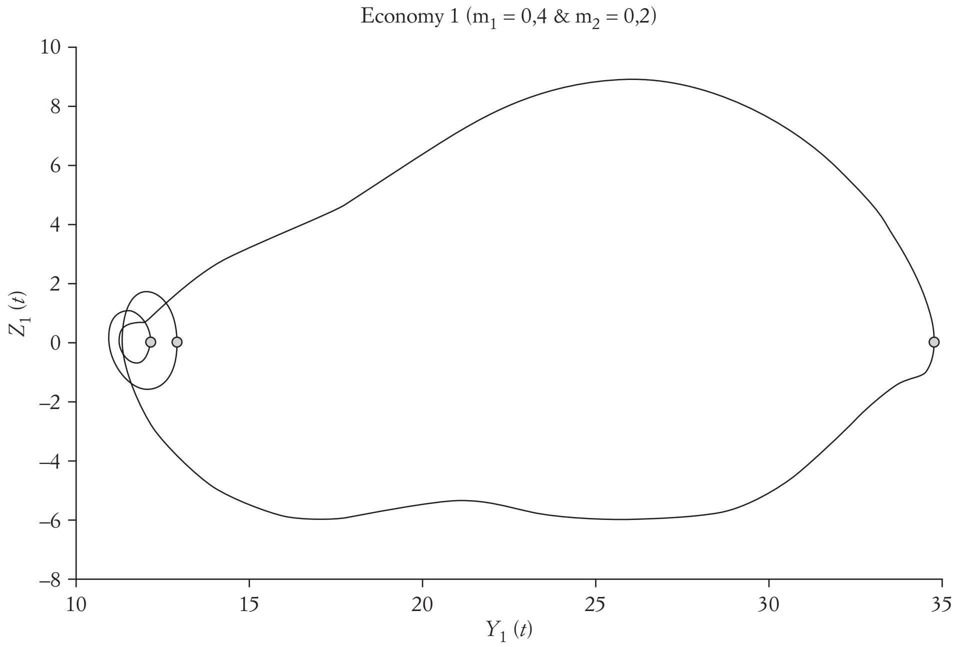


FIG. 2

$$\Omega_2(t) = \frac{C_{02} - e_2 Y_2(t) - a_2 Z_2(t) + m_1(Y_1(t) + f_2 Z_1(t)) + \varphi_2(v_2 Y_2(t) - K_2(t))}{\vartheta_2 \varepsilon_2}$$

$$\omega = \frac{\varepsilon_1 \varepsilon_2}{\varepsilon_1 \varepsilon_2 + m_1 m_2 (\varepsilon_1 - \varepsilon_2)}$$

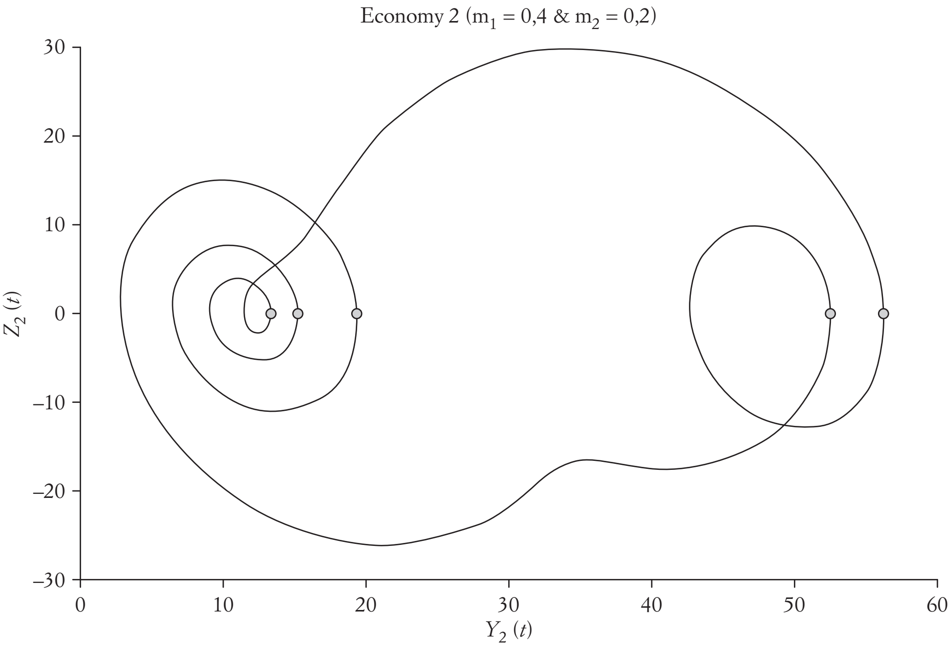
$$f_1 = \vartheta_1 + (\varepsilon_1 - \varepsilon_2)$$

$$f_2 = \vartheta_2 + (\varepsilon_2 - \varepsilon_1)$$

$$e_1 = (1 + m_1 - c_1)$$

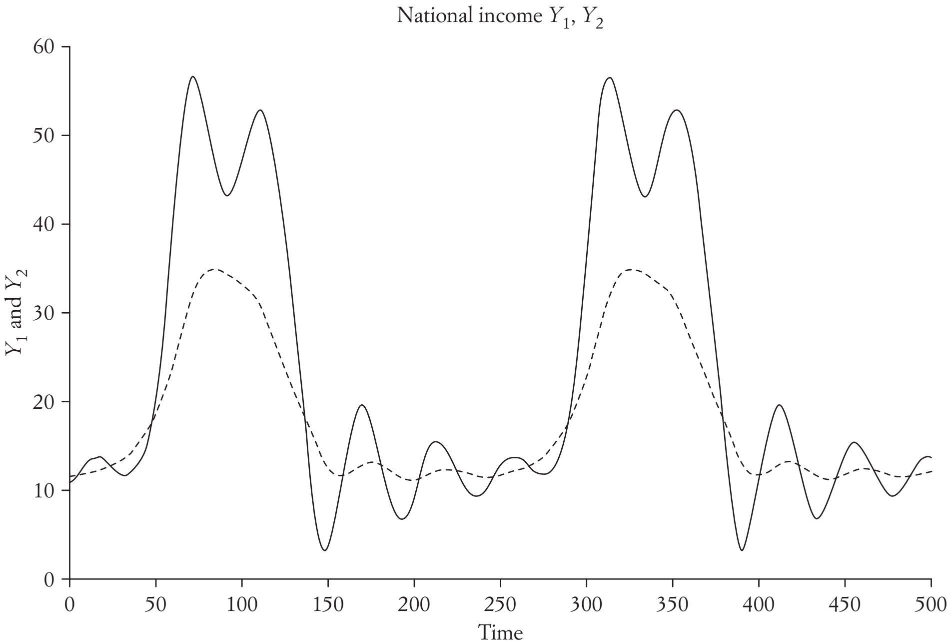
$$e_2 = (1 + m_2 - c_2)$$

The above is a six dimensional dynamical system composed of two coupled oscillators (economies). For certain sets of the parameters the two economies will be highly synchronized, while for others they are highly asynchronized.



Economy 2 (m_1 = 0,4 & m_2 = 0,2)
30
20
10
0
−10
−20
−30
Z_2 (t)
0
10
20
30
40
50
60
Y_2 (t)

FIG. 3



National income Y_1, Y_2
60
50
40
30
20
10
0
Y_1 and Y_2
0
50
100
150
200
250
300
350
400
450
500
Time

FIG. 4

6. Concluding Note: Digital and Analog Computing

In this paper I have made a conjecture that the Phillips MONIAC Machine is best modeled as a nonlinear system of difference-differential equations (eqs. 9, 10, 11), which is in turn approximated with the nonlinear differential equation 12 which has state space representation (eq. 13).

It would be interesting to investigate whether the above approximated digital simulations do, for appropriately chosen parameters, shadow the original Phillips analogue machine. Here, I have gone further and I have tried to simulate, as suggested by Phillips at the end of his article (1950, p. 306), the behavior of two coupled virtual Phillips machines. To my knowledge, this exercise has not been done before. It would be interesting toperform the same exercise on two or more coupled actual (non-virtual) Phillips machines.

The richness and the specific dynamic behavior of coupled dynamical systems depend on several factors. The theoretical system (15) for two countries, which is generalized to the n-countries in Zambelli (2011) are Taylor series approximations of the original theoretical model. Moreover in order to conduct digital computations, the approximated systems are further approximated with difference equations; different approximating algorithms: Euler, Runge-Kutta and so on.

Before concluding, I would like to point out that, here, I have made the conjecture that the mathematical description that Phillips (1950) provided for the functioning of his machine, being linear, is inappropriate. I have proposed the Goodwin flexible-accelerator model as a better mathematical formulation of the MONIAC. But, this formulation is also inadequate. As explained above, a better mathematical formulation seems to be in terms of a difference-differential nonlinear system of equations. The computation of the dynamics of such a system is best made with an analogue computer like the Phillips MONIAC or with an analogue electrical system like the one constructed by Strotz *et al.* (1953).

These analogue systems are extremely useful to study complex dynamical systems without relying on «linear» first approximations. To do that we might need the use of (analog) computing machines and, perhaps, we need to study the Phillips MONIAC machine itself. In the concluding section of *Dynamical Coupling with Especial Reference to Markets Having Production Lags*, Goodwin (1947, p. 204) wrote that there are «[...] complications arising with general dynamic interdependence. To go from two identical markets to n non-identical ones will require the prolonged services of yet unborn calculating machines». Whether Goodwin meant digital or analogue calculating machines is not known. Goodwin and Phillips were friends and have surely discussed these matters.

Appendix

Here are reported the parameter levels associated with the generations of the different graphs.

Figures 2, 3, 4

$$\vartheta_1 = 1,\ \ \varepsilon_1 = 0.25,\ \ C_{01} = 10,\ \ c_1 = 0.6,\ \ v_1 = 2.0,\ \ k_1^* = -3,\ \ k_1^{**} = 9,$$
$$\vartheta_2 = 0.5,\ \ \varepsilon_2 = 0.25,\ \ C_{02} = 2,\ \ c_1 = 0.8,\ \ v_2 = 1.4,\ \ k_1^* = -2,\ \ k_2^{**} = 4,$$

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Summary: Coupled Dynamics in a Phillips Machine Model of the Macroeconomy (J.E.L. A10, B16, B41, C63, F44)

In this paper it is claimed that the Phillips machine, contrary to what is sometimes believed, is a nonlinear mechanism. Phillips (1950) presented his machines, as formally described, with linear differential equations. In doing so, he fell into what Samuelson came later to define as the *linearity dogma*. It is argued here that the Phillips machines are not linear mechanisms but are coherent with a system of nonlinear difference-differential equations. Here, this system is in turn approximated with a set of nonlinear differential equations, which are isomorphic with the Hicks-Goodwin flexible multiplier accelerator models. Consistent with this conjecture and following a Goodwin-Phillips' suggestion, we have presented a «digital» simulation of two coupled would-be Phillips machines. Whether the data generated with the set of nonlinear difference equations is consistent with the observations emerging from the actual coupled Phillips machines could be verified by comparing the data emerging from the two systems – the digital and the analogue ones.